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The glassy phase of the complex branching Brownian motion energy model*

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Abstract

We identify the fluctuations of the partition function for a class of random energy models, where the energies are given by the positions of the particles of the complex-valued branching Brownian motion (BBM). Specifically, we provide the weak limit theorems for the partition function in the so-called "glassy phase" – the regime of parameters, where the behaviour of the partition function is governed by the extrema of BBM. We allow for arbitrary correlations between the real and imaginary parts of the energies. This extends the recent result of Madaule, Rhodes and Vargas [19], where the uncorrelated case was treated. In particular, our result covers the case of the real-valued BBM energy model at complex temperatures.

Keywords: Gaussian processes; branching Brownian motion; logarithmic correlations; random energy model; phase diagram; extremal processes; cluster processes; multiplicative chaos.
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1 Introduction

Phase transitions arise via an analyticity breaking of the logarithm of the partition function (see, e.g., Ruelle [22]). To analyse this phenomenon, the study of partition functions at *complex temperatures* is of a key interest, as was observed by Lee and Yang [24, 17]. Another motivation to study complex-valued Hamiltonians comes from quantum physics. There, partition functions with complex energies emerge naturally, e.g., from the Schrödinger equation via "imaginary time" Feynman's path integrals.

It is believed that large classes of models of disordered systems fall in the same universality class and, in particular, share the same shape of the phase diagram. Random energy models were proven to be useful in exploring universality classes in mean-field disordered systems, see, e.g., Bovier [6], Panchenko [21] and Kistler [13]. A number of random energy models with complex energies has been considered in the literature. One of the simplest such models (in terms of the correlation structure of the energies) is the so called *Random Energy Model* (REM). For this model, the analyticity of the log-partition function was studied in the seminal work by Derrida [9] and later by

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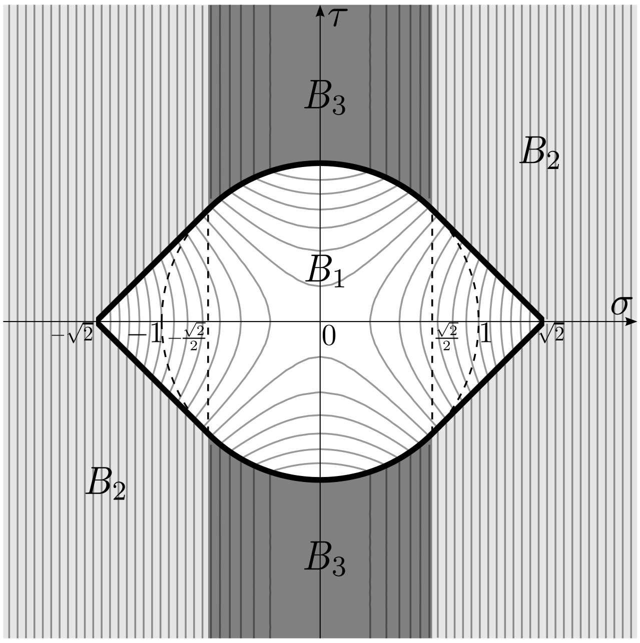

Figure 1: Phase diagram of the REM (and conjecturally of the BBM energy model). The grey curves are the level lines of the limiting log-partition function, cf. (1.18). This paper mainly deals with phase B_2.

Koukiou [15]. The full phase diagram of this model at complex temperatures including the fluctuations and zeros of the partition function were identified by Kabluchko and one of us in [11]. In particular, the case of arbitrary correlations between the imaginary and real parts of the energies was considered in [11]. The same authors answered in [12] similar questions about the *Generalized Random Energy model* (GREM) – a model with hierarchical correlations – and obtained the full phase diagram. In the complex GREM, the phase diagram turned out to have a much richer structure than that of the complex REM. This sheds some light on the phase diagrams of the models beyond the complex REM universality class.

It is known that models with *logarithmic correlations* between the energies are at the borderline of the REM universality class. In particular, they are expected to have the same phase diagram. This has been shown for directed polymers on a tree with complex-valued energies by Derrida, Evans, and Speer [10], and for a model of complex multiplicative cascades by Barral, Jin, and Mandelbrot [5]. Lacoin, Rhodes, and Vargas [16] analysed the phase diagram for complex *Gaussian multiplicative chaos* – a model with logarithmic correlations between the energies on a Euclidean space. There, only the case without correlations between the imaginary and real parts of the energy was treated. It turned out that the phase diagram coincides with the REM one, see Figure 1.

In [16], the analysis of the so-called "glassy" phase B_2, see Figure 1, was left open. In this phase, the partition function is dominated by the extreme values of the energies. Phase B_2 was analysed by Madaule, Rhodes, and Vargas [19] in a continuous model with logarithmic correlations on a tree – the complex *BBM energy model*, but again only when the imaginary and real parts of the energies are uncorrelated. In this model, a deeper understanding of phase B_2 is possible due to recent progress in the analysis of the extremal process of BBM by Aïdékon, Berestycki, Brunet, and Shi [1] and Arguin, Bovier, and Kistler [3]. Madaule, Rhodes, and Vargas [20], have recently analysed the behaviour of the partition function on the boundary between phases B_1 and B_2 (see Figure 1).

In this article, we extend the result of [19]. Specifically, we prove the weak conver-

gence of the (rescaled) partition function of the complex BBM energy model in phase B_2 to a non-trivial distribution. We allow for arbitrary correlations between the real and imaginary parts of the energy. In particular, this covers the complex temperature case, in which the real and imaginary parts of the random energies have maximal correlation (i.e., they are a.s. equal). This case is especially relevant for the Lee-Yang program.

1.1 Branching Brownian motion.

Before stating our results, let us briefly recall the construction of a BBM. Consider a canonical continuous branching process: a *continuous time Galton-Watson* (GW) process [4]. It starts with a single particle at time zero. After an exponential time of parameter one, this particle splits into $k \in \mathbb{Z}_+$ particles according to some probability distribution $(p_k)_{k\geq 0}$ on $\mathbb{Z}_+$. Then, each of the new-born particles splits independently at independent exponential (parameter 1) times again according to the same $(p_k)_{k\geq 0}$, and so on. We assume that $\sum_{k=1}^{\infty} p_k = 1$.[1] In addition, we assume that $\sum_{k=1}^{\infty} kp_k = 2$ (i.e., the expected number of children per particle equals two)[2]. Finally, we assume that $K := \sum_{k=1}^{\infty} k(k-1)p_k < \infty$ (finite second moment)[3]. At time $t = 0$, the GW process starts with just one particle.

For given $t \geq 0$, we label the particles of the process as $i_1(t), \dots, i_{n(t)}(t)$, where $n(t)$ is the total number of particles at time t. Note that under the above assumptions, we have $\mathbb{E}[n(t)] = \mathrm{e}^t$. For $s \leq t$, we denote by $i_k(s,t)$ the unique ancestor of particle $i_k(t)$ at time s. In general, there will be several indices k, l such that $i_k(s,t) = i_l(s,t)$. For $s, r \leq t$, define the time of the most recent common ancestor of particles $i_k(r,t)$ and $i_l(s,t)$ as

$$d(i_k(r,t), i_l(s,t)) := \sup\{u \leq s \wedge r \colon i_k(u,t) = i_l(u,t)\}. \tag{1.1}$$

For $t \geq 0$, the collection of all ancestors naturally induces the random tree

$$\mathbb{T}_t := \{i_k(s,t) \colon 0 \leq s \leq t, 1 \leq k \leq n(t)\} \tag{1.2}$$

called the *GW tree up to time* t. We denote by $\mathcal{F}^{\mathbb{T}_t}$ the σ-algebra generated by the GW process up to time t.

In addition to the genealogical structure, the particles get a *position* in $\mathbb{R}$. Specifically, the first particle starts at the origin at time zero and performs Brownian motion until the first time when the GW process branches. After branching, each new-born particle independently performs Brownian motion (started at the branching location) until their respective next branching times, and so on. We denote the positions of the $n(t)$ particles at time $t \geq 0$ by $x_1(t), \dots, x_{n(t)}(t)$ and by $x_1(s,t), \dots, x_{n(t)}(s,t)$ the positions of their ancestors at time $s \geq 0$.

We define BBM as a family of Gaussian processes,

$$x_t := \{x_1(s,t), \dots, x_{n(t)}(s,t) \colon s \leq t\} \tag{1.3}$$

indexed by time horizon $t \geq 0$. Note that conditionally on the underlying GW tree these Gaussian processes have the following covariance

$$\mathbb{E}\left[x_k(s,t)x_l(r,t) \mid \mathcal{F}^{\mathbb{T}_t}\right] = d(i_k(s,t), i_l(r,t)), \quad s, r \in [0,t], \quad k, l \leq n(t). \tag{1.4}$$

[1] This implies that $p_0 = 0$, so none of the particles ever dies.

[2] The latter assumption is just a matter of normalization. Any expected number of children greater than 1 (= the supercritical regime) is allowed and the results of this paper remain valid with appropriate modifications of constants.

[3] Under the stated conditions, the convergence of the extremal process of BBM, on which we rely, is proven in [3]. For the case of branching random walk, using truncation techniques, Madaule [18] has shown the same under conditions that would in the Gaussian case imply finiteness of $\sum_k p_k k (\ln k)^3$. This could probably be carried over to BBM. It is not clear whether the result holds under the Kesten-Stigum condition $\sum_k p_k k \ln k < \infty$. For a discussion on these issues, we refer to the lecture notes by Shi [23]. In the present paper, we are not concerned with improving the conditions on the offspring distribution.

Bramson [7, 8] showed that

$$m(t) := \sqrt{2}t - \frac{3}{2\sqrt{2}} \log t \tag{1.5}$$

is the order of the maximal position among all BBM particles alive at large time t, i.e.,

$$\lim_{t \uparrow \infty} \mathbb{P} \left\{ \max_{k \le n(t)} x_k(t) - m(t) \le y \right\} = \mathbb{E} \left[\mathrm{e}^{-CZ\mathrm{e}^{-\sqrt{2}y}} \right], \quad y \in \mathbb{R}, \tag{1.6}$$

where $C > 0$ is a constant and Z is the a.s. limit of the so-called *derivative martingale*:

$$Z := \lim_{t \uparrow \infty} \sum_{k=1}^{n(t)} (\sqrt{2}t - x_k(t)) \mathrm{e}^{-\sqrt{2}(\sqrt{2}t - x_k(t))}, \quad \text{a.s.} \tag{1.7}$$

In [1, 3], as $t \uparrow \infty$, the non-trivial limiting point process of the (shifted by $m(t)$) particles of BBM was identified. Specifically, it was shown that the point process,

$$\mathcal{E}_t := \sum_{k=1}^{n(t)} \delta_{x_k(t) - m(t)}, \quad t \in \mathbb{R}_+ \tag{1.8}$$

converges in law as $t \uparrow \infty$ to the point process

$$\mathcal{E} := \sum_{k,l} \delta_{\eta_k + \Delta_l^{(k)}}, \tag{1.9}$$

where:

(a) $\{\eta_k\}_{k \in \mathbb{N}} \subset \mathbb{R}$ are the atoms of a Cox process with *random intensity measure* $CZ\mathrm{e}^{-\sqrt{2}y}\mathrm{d}y$, where C and Z are the same as in (1.6).

(b) $\{\Delta_l^{(k)}\}_{l \in \mathbb{N}} \subset \mathbb{R}$ are the atoms of independent and identically distributed point processes $\Delta^{(k)}$, $k \in \mathbb{N}$ called *clusters* which are independent copies of the limiting point process

$$\Delta := \lim_{t \uparrow \infty} \sum_{k=1}^{n(t)} \delta_{\hat{x}_k(t) - \max_{l \le n(t)} \hat{x}_l(t)} \tag{1.10}$$

with $\hat{x}(t)$ being BBM $x(t)$ conditioned on $\max_{k \le n(t)} x_k(t) \ge \sqrt{2}t$.

1.2 Branching Brownian motion energy model at complex temperatures with arbitrary correlations

Let $\rho \in [-1, 1]$. For any $t \in \mathbb{R}_+$, let $X(t) := (x_k(t))_{k \le n(t)}$ and $Y(t) := (y_k(t))_{k \le n(t)}$ be two BBMs with the same underlying GW tree such that, for $k \le n(t)$,

$$\mathrm{Cov}(x_k(t), y_k(t)) = |\rho| t. \tag{1.11}$$

Then,

$$Y(t) \stackrel{\mathrm{D}}{=} \rho X(t) + \sqrt{1 - \rho^2} Z(t), \tag{1.12}$$

where "$\stackrel{\mathrm{D}}{=}$" denotes equality in distribution and $Z(t) := (z_i(t))_{i \le n(t)}$ is a branching Brownian motion with the same underlying GW process which is independent from $X(t)$. Representation (1.12) allows us to handle arbitrary correlations by decomposing the process Y into a part independent from X and a fully correlated one.

We define the partition function for the complex BBM energy model with correlation ρ at inverse temperature $\beta := \sigma + i\tau \in \mathbb{C}$ by

$$\widetilde{\mathcal{X}}_{\beta,\rho}(t) := \sum_{k=1}^{n(t)} \mathrm{e}^{\sigma x_k(t) + i\tau y_k(t)}. \tag{1.13}$$

1.3 Main results

Let us specify the three phases depicted on Figure 1 analytically:

$$\begin{aligned} B_1 := \mathbb{C} \setminus \overline{B_2 \cup B_3}, \quad & B_2 := \{\sigma + i\tau \in \mathbb{C} \colon 2\sigma^2 > 1, |\sigma| + |\tau| > \sqrt{2}\}, \\ & B_3 := \{\sigma + i\tau \in \mathbb{C} \colon 2\sigma^2 < 1, \sigma^2 + \tau^2 > 1\}. \end{aligned} \tag{1.14}$$

In this paper, we focus on the *glassy phase* B_2. We start with the convergence of the partition function in the case of the real BBM energy model at complex temperatures. We say that a complex-valued r.v. Y is *isotropic α-stable* if there exists $c \in \mathbb{R}_+$ and $\alpha \in (0, 2]$ such that

$$\mathbb{E}[\mathrm{e}^{i\mathrm{Re}(\bar{z}Y)}] = \mathrm{e}^{-c|z|^\alpha}, \quad \text{for all } z \in \mathbb{C}. \tag{1.15}$$

Recall the notation from (1.9).

Theorem 1.1 (Partition function fluctuations for $|\rho| = 1$)**.** *For $\beta = \sigma + i\tau \in B_2$, the rescaled partition function $\mathcal{X}_{\beta,1}(t) := \mathrm{e}^{-\beta m(t)} \widetilde{\mathcal{X}}_{\beta,1}(t)$ converges in law to the r.v.*

$$\mathcal{X}_{\beta,1} := \sum_{k,l \geq 1} \mathrm{e}^{\beta\left(\eta_k + \Delta_l^{(k)}\right)}, \quad \textit{as } t \uparrow \infty. \tag{1.16}$$

Theorem 1.2 (Partition function fluctuations for $|\rho| \in (0, 1)$)**.** *For $\beta = \sigma + i\tau \in B_2$ and $|\rho| \in (0, 1)$, the rescaled partition function $\mathcal{X}_{\beta,\rho}(t) := \mathrm{e}^{-\sigma m(t)} \widetilde{\mathcal{X}}_{\beta,\rho}(t)$ converges in law to the r.v. $\mathcal{X}_{\beta,\rho}$, as $t \uparrow \infty$. Conditionally on Z, $\mathcal{X}_{\beta,\rho}$ is a complex isotropic $\sqrt{2}/\sigma$-stable r.v.*

Remark 1.3. For $\rho = 0$, Theorem 1.2 was proven in [19]. Our proof uses a representation of correlated real and imaginary parts in terms of independent BBM's. As in [19], we control second moments. However, the way we do this is different and simpler then the method used in that paper, which relies on decomposing the paths of the BBM particles according to the time and location of the minimal position along the given path. Our approach uses instead the upper envelope for ancenstral paths that was obtained in [2].

Remark 1.4. Note that the fluctuations of the partition function in the complex BBM energy model (cf., Theorems 1.1, 1.2) are governed by the extremal process $\mathcal{E}$. Thus, the fluctuations are different from the ones in the complex REM [11, Theorems 2.8, 2.20] which are governed by a Poisson point process. Despite the differences in fluctuations, we conjecture that in the limit as $t \uparrow \infty$ the *log-partition function*

$$p_t(\beta) := \frac{1}{t} \log |\widetilde{\mathcal{X}}_{\beta,\rho}(t)|, \quad t \in \mathbb{R}_+, \quad \beta \in \mathbb{C} \tag{1.17}$$

of the complex BBM energy model is the same as in the complex REM.

Conjecture 1.5 (Phase diagram)**.** *For any $\rho \in [-1, 1]$, the complex BBM energy model has the same free energy and the phase diagram (cf.,* Figure 1*) as the complex REM, i.e.,*

$$\lim_{t \uparrow \infty} p_t(\beta) =: p(\beta) = \begin{cases} 1 + \frac{1}{2}(\sigma^2 - \tau^2), & \beta \in \overline{B_1}, \\ \sqrt{2}|\sigma|, & \beta \in \overline{B_2}, \\ \frac{1}{2} + \sigma^2, & \beta \in \overline{B_3}, \end{cases} \tag{1.18}$$

and the convergence in (1.18) *holds in probability and in L^1.*

Remark 1.6. Convergence in probability for $\beta \in B_2$ in (1.18) follows from Theorems 1.1 and 1.2 by [11, Lemma 3.9 (1)]. The remaining Parts B_1 and B_3 of Conjecture 1.5 are supported by results for similar models, e.g., [10, 5, 16, 11, 12] and by the following intuition.

For $\beta \in B_1$, $\widetilde{\mathcal{X}}_{\beta,\rho}(t)/\mathbb{E}[\widetilde{\mathcal{X}}_{\beta,\rho}(t)]$ is an L^1-convergent complex-valued martingale (as $t \to \infty$) with expectation 1 and a simple computation shows that

$$|\mathbb{E}[\widetilde{\mathcal{X}}_{\beta,\rho}(t)]| = \exp\left(t + \frac{1}{2}t(\sigma^2 - \tau^2)\right). \tag{1.19}$$

See Appendix A for the L^2-martingale convergence in the domain $|\beta| < 1$.

For $\beta \in B_3$, the variance of the partition function of the REM with e^t independent particles equals

$$\mathrm{e}^t \left(\mathbb{E}[\exp(2\sigma x_1(t))] - \exp\left(\frac{1}{2}t(\sigma^2 - \tau^2)\right)\right) \underset{t\uparrow\infty}{\sim} \exp\left(t + 2\sigma^2 t\right), \tag{1.20}$$

cf. [11]. Therefore, as $t \uparrow \infty$, the standard deviation has a greater order of magnitude than the expectation (1.19). So, in view of the central limit theorem, it is plausible that

$$\widetilde{\mathcal{X}}_{\beta,\rho}(t)/\exp\left(\frac{1}{2}t + \sigma^2 t\right) \tag{1.21}$$

converges as $t \uparrow \infty$ in distribution. However, due to correlations between the particle positions of BBM, the limiting distribution in (1.21) need not be Gaussian, cf. [16, Theorems 4.2 and 6.6] and [11, Eq. (2.11)].

Organization of the rest of the paper. The proofs of Theorems 1.1 and 1.2 consist of two main steps. First, we show that only the extremal particles can contribute to the partition function in the limit as $t \uparrow \infty$ (cf., Proposition 2.1 and its proof in Section 3). Second, we use the continuous mapping theorem to deduce Theorems 1.1 and 1.2 from the behaviour of the extremal process. This is done in Section 2.

2 Convergence of the partition function

First, we state that in the glassy phase B_2 only the extremal particles can contribute to the limit of the partition function as t tends to infinity.

Proposition 2.1. *If $|\rho| \in (0, 1]$ and $\beta \in B_2$, then, for all $\delta, \epsilon > 0$, there exists $A_0 > 0$ such that, for all $A > A_0$ and all t sufficiently large,*

$$\mathbb{P}\left\{\left|\sum_{k=1}^{n(t)} \mathrm{e}^{\sigma(x_k(t)-m(t))+i\tau y_k(t)} \mathbb{1}_{\{x_k(t)-m(t)<-A\}}\right| > \delta\right\} < \epsilon. \tag{2.1}$$

The proof of Proposition 2.1 is postponed until Section 3. Using Proposition 2.1 together with the continuous mapping theorem, we now prove Theorem 1.1.

Proof of Theorem 1.1. Denote by $\mathbb{M}$ the space of locally finite counting measures on $\overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$. We endow $\mathbb{M}$ with the vague topology. Consider for $A \in \mathbb{R}_+$ the functional $\Phi_{\beta,A} \colon \mathbb{M} \to \mathbb{R}$. This functional maps a locally finite counting measure $\zeta = \sum_{i\in I} \delta_{x_i}$ to $\Phi_{\beta,A}(\zeta) := \sum_{i\in I} \mathrm{e}^{\beta x_i} \mathbb{1}_{\{x_i > -A\}}$, where I is a countable index set. The set of locally finite measures ζ on which the functional $\Phi_{\beta,A}$ is not continuous (i.e., ζ charging $-A$ or $+\infty$) has zero measure w.r.t. the law of $\mathcal{E}$. Hence, by the continuous mapping theorem, it follows that $\Phi_{\beta,A}(\mathcal{E}_t)$ converges in law to $\Phi_{\beta,A}(\mathcal{E})$, which is equal to

$$\sum_{k,l\geq 1} \mathrm{e}^{\beta\left(\eta_k + \Delta_l^{(k)}\right)} \mathbb{1}_{\{\eta_k + \Delta_l^{(k)} \geq -A\}}. \tag{2.2}$$

Note that by Proposition 2.1, for all $\epsilon > 0$ and $\delta > 0$, there exists A_0 such that, for all $A > A_0$ and all t sufficiently large,

$$\mathbb{P}\left\{|\mathcal{X}_{\beta,1}(t) - \Phi_{\beta,A}(\mathcal{E}_t)| > \delta\right\} < \epsilon. \tag{2.3}$$

Hence, by Slutsky's Theorem (see, e.g., [14, Theorem 13.18]), $\mathcal{X}_{\beta,1}(t)$ converges in law to

$$\lim_{A\uparrow\infty}\sum_{k,l\geq 1} e^{\beta\left(\eta_k+\Delta_l^{(k)}\right)}\mathbb{1}_{\{\eta_k+\Delta_l^{(k)}\geq -A\}} \tag{2.4}$$

which is equal to $\mathcal{X}_{\beta,1}$. □

We now prove Theorem 1.2.

Proof of Theorem 1.2. Using Representation (1.12), we have that $\mathcal{X}_{\beta,\rho}(t)$ is in distribution equal to

$$\sum_{k=1}^{n(t)} \mathrm{e}^{(\sigma+i\rho\tau)(x_k-m(t))+i\sqrt{1-\rho^2}\tau z_k(t)-i\rho\tau m(t)}, \tag{2.5}$$

where $(z_k(t), k\leq n(t))$ are the particles from a BBM that is independent from $X(t)$ (but with respect to the same GW tree). If $|\rho|\neq 1$, then by [19, see Lemma 3.2 and the subsequent discussion before Eq. (3.7) therein] we get that

$$G(t):=\sum_{k=1}^{n(t)}\delta_{(x_k(t)-m(t),\exp(i\sqrt{1-\rho^2}\tau z_k(t)-i\rho\tau m(t)))} \tag{2.6}$$

converges weakly as $t\uparrow\infty$ to

$$\mathcal{G}:=\sum_{k,l\geq 1}\delta_{(p_k+\Delta_l^{(k)},U^{(k)}\widetilde{W}_l^{(k)})}, \tag{2.7}$$

where $\left(U^{(k)}\right)_{k\geq 1}$ are i.i.d. uniformly distributed on the unit circle and $\widetilde{W}_l^{(k)}$ are the atoms of a point process on the unit circle. The description of $\widetilde{W}^{(k)}$ could be made more explicit using the description of the cluster process Δ obtained in [1, Theorem 2.3] that encodes the genealogical structure of Δ.

Denote by $\widetilde{\mathbb{M}}$ the space of locally finite counting measures on $\overline{\mathbb{R}}\times\{z\in\mathbb{C}\colon |z|=1\}$. We endow $\widetilde{\mathbb{M}}$ with the (Polish) topology of vague convergence. For $A\in\mathbb{R}_+$, consider the functional $\widetilde{\Phi}_{\beta,A}\colon\widetilde{\mathbb{M}}\to\mathbb{C}$ that maps a locally finite counting measure $\tilde{\zeta}=\sum_{k\in I}\delta_{(x_k,z_k)}$ to $\widetilde{\Phi}_{\beta,A}(\zeta):=\sum_{k\in I}\mathrm{e}^{\beta x_k}z_k\mathbb{1}_{\{x_k>-A\}}$, where I is a countable index set. The set of locally finite measures ζ on which the functional $\Phi_{\beta,A}$ is not continuous (i.e., $\widetilde{\zeta}$ charging $(-A,\cdot)$ or $(+\infty,\cdot)$) has zero measure w.r.t. the law of $\mathcal{G}$. Hence, by the continuous mapping theorem, it follows that $\widetilde{\Phi}_{\sigma+i\rho\tau,A}(\mathcal{G}_t)$ converges in law to $\widetilde{\Phi}_{\sigma+i\rho\tau,A}(\mathcal{G})$, which is equal to

$$\sum_{k,l\geq 1}\mathrm{e}^{(\sigma+i\rho\tau)\left(\eta_k+\Delta_l^{(k)}\right)}U^{(k)}\widetilde{W}_l^{(k)}\mathbb{1}_{\{\eta_i+\Delta_l^{(k)}\geq -A\}}. \tag{2.8}$$

Since $\mathrm{e}^{(i\rho\tau)\left(\eta_k+\Delta_l^{(k)}\right)}U^{(k)}$ is also uniformly distributed on the unit circle, (2.8) is equal in distribution to

$$\sum_{k,l\geq 1}\mathrm{e}^{\sigma\left(\eta_k+\Delta_l^{(k)}\right)}U^{(k)}\widetilde{W}_l^{(k)}\mathbb{1}_{\{\eta_i+\Delta_l^{(k)}\geq -A\}}. \tag{2.9}$$

Note that again by Proposition 2.1, for all $\epsilon>0$ and $\delta>0$, there exists A_0 such that, for all $A>A_0$ and all t sufficiently large,

$$\mathbb{P}\left\{\left|\mathcal{X}_{\beta,\rho}(t)-\widetilde{\Phi}_{\sigma+i\rho\tau,A}(\mathcal{G}_t)\right|>\delta\right\}<\epsilon. \tag{2.10}$$

Hence, by Slutsky's theorem (see, e.g., [14, Theorem 13.18]), $\mathcal{X}_{\beta,\rho}(t)$ converges in law to

$$\lim_{A\uparrow\infty}\sum_{k,l\geq 1}\mathrm{e}^{\sigma\left(\eta_k+\Delta_l^{(k)}\right)}U^{(k)}\widetilde{W}_l^{(k)}\mathbb{1}_{\{\eta_k+\Delta_l^{(k)}\geq -A\}}=\sum_{k,l\geq 1}\mathrm{e}^{\sigma\left(\eta_k+\Delta_l^{(k)}\right)}U^{(k)}\widetilde{W}_l^{(k)}. \tag{2.11}$$

We rewrite (2.11) as

$$\sum_{k\geq 1}\mathrm{e}^{\sigma\eta_k}U^{(k)}W^{(k)}, \tag{2.12}$$

where $W^{(k)}:=\sum_l \mathrm{e}^{\sigma\Delta_l^{(k)}}\widetilde{W}_l^{(k)}$, $k\geq 1$ are i.i.d. r.v.'s. From (2.12), it follows that conditionally on Z, the distribution of $\mathcal{X}_{\beta,\rho}$ is complex isotropic $\sqrt{2}/\sigma$-stable.

□

3 Proof of Proposition 2.1

Due to symmetry, we only prove Proposition 2.1 for $\sigma,\tau>0$. In the proof of Proposition 2.1, we distinguish two cases:

$$\textbf{(a)}\quad \sigma>\sqrt{2};\qquad \textbf{(b)}\quad \sqrt{2}/2<\sigma\leq\sqrt{2} \text{ and } \sigma+\tau>\sqrt{2}. \tag{3.1}$$

Case (a). In this case, the proof works as in the independent case treated in [19, Lemma 3.5]. For completeness, we also provide the proof in this case. We use a first moment computation together with the upper bound on the maximal position of all particles obtained in [2, Theorem 2.2].

Proof of Proposition 2.1 *in case* (a). Recall the notation from (1.3). By [2, Theorem 2.2], for $0<\gamma<\frac{1}{2}$, there exists $r_\epsilon>0$ such that for all $r>r_\epsilon$ and $t>3r$

$$\mathbb{P}\left\{\exists k\leq n(t)\colon x_k(s,t)>U_{t,\gamma} \text{ for some } s\in[r,t-r]\right\}<\frac{\epsilon}{2}, \tag{3.2}$$

where $U_{t,\gamma}(s):=\frac{s}{t}m(t)+(s\wedge(t-s))^\gamma$. Define the following set on the path space

$$\mathcal{U}_{t,r,\gamma}:=\{x(\cdot)\in C(\mathbb{R}_+,\mathbb{R})\colon x(s,t)\leq\frac{s}{t}m(t)+(s\wedge(t-s))^\gamma, \forall s\in[r,t-r]\}. \tag{3.3}$$

By (3.2), to show (2.1), it suffices to check that, for sufficiently large $A>0$,

$$\mathbb{P}\left\{\Big|\sum_{k=1}^{n(t)}\mathrm{e}^{\sigma(x_k(t)-m(t))+i\tau y_k(t)}\mathbb{1}_{\{x_k(t)-m(t)<-A\}\cap\{x_k\in\mathcal{U}_{t,r,\gamma}\}}\Big|>\delta\right\}<\epsilon/2. \tag{3.4}$$

By Markov's inequality, the probability in (3.4) is bounded from above by

$$\begin{aligned}&\frac{1}{\delta}\mathbb{E}\left[\left|\sum_{k=1}^{n(t)}\mathrm{e}^{\sigma(x_k(t)-m(t))+i\tau y_k(t)}\mathbb{1}_{\{x_k(t)-m(t)<-A\}\cap\{x_k\in\mathcal{U}_{t,r,\gamma}\}}\right|\right]\\&\leq\frac{1}{\delta}\mathbb{E}\left[\sum_{k=1}^{n(t)}\mathrm{e}^{\sigma(x_k(t)-m(t))}\mathbb{1}_{\{x_k(t)-m(t)<-A\}\cap\{x_k\in\mathcal{U}_{t,r,\gamma}\}}\right].\end{aligned} \tag{3.5}$$

We rewrite the expectation in the r.h.s. of (3.5) as $\sum_{B>A}S(B,t)$, where

$$S(B,t):=\mathbb{E}\left[\sum_{k=1}^{n(t)}\mathrm{e}^{\sigma(x_k(t)-m(t))}\mathbb{1}_{\{x_k(t)-m(t)\in(-B+1,-B]\}\cap\{x_k\in\mathcal{U}_{t,r,\gamma}\}}\right]. \tag{3.6}$$

Next, we manipulate the event

$$\begin{aligned}&\{x_k(t)-m(t)\in(-B+1,-B]\}\cap\{x_k\in\mathcal{U}_{t,r,\gamma}\}\\&\subset\{x_k(t)-m(t)\in(-B+1,-B]\}\cap\{\xi(s)\leq\frac{s}{t}B+(s\wedge(t-s))^\gamma,\forall s\in[r,t-r]\},\end{aligned} \tag{3.7}$$

where $\xi_k(s):=x_k(s,t)-\frac{s}{t}x_k(t)$ is a Brownian bridge from 0 to 0 in time t that is independent from $x_k(t)$. Hence, we can bound $S(B,t)$ from above by

$$\begin{aligned}&\mathbb{E}\left[\sum_{k=1}^{n(t)}\mathrm{e}^{\sigma(x_k(t)-m(t))}\mathbb{1}_{\{x_k(t)-m(t)\in(-B+1,-B]\}\cap\{\xi_k(s)\leq\frac{s}{t}B+(s\wedge(t-s))^\gamma,\forall s\in[r,t-r]\}}\right] \\ &=\mathrm{e}^t\mathbb{E}\left[\mathrm{e}^{\sigma(x(t)-m(t))}\mathbb{1}_{x(t)-m(t)\in(-B+1,-B]}\right]\mathbb{P}\left\{\xi(s)\leq\frac{s}{t}B+(s\wedge(t-s))^\gamma,\forall s\in[r,t-r]\right\},\end{aligned} \tag{3.8}$$

where $x(t)$ is normal distributed with mean 0 and variance t and $\xi(\cdot)$ is a Brownian bridge from 0 to 0 in time t independent from $x(t)$. The expectation in the second line of (3.8) is equal to

$$\int_{m(t)-B}^{m(t)-B+1}\mathrm{e}^{\sigma(x-m(t))}\mathrm{e}^{-x^2/2t}\frac{\mathrm{d}x}{\sqrt{2\pi t}}=\mathrm{e}^{-\sigma m(t)+\frac{\sigma^2t}{2}}\int_{m(t)-B-\sigma t}^{m(t)-B+1-\sigma t}\mathrm{e}^{-w^2/2t}\frac{\mathrm{d}w}{\sqrt{2\pi t}}, \tag{3.9}$$

where we changed variables $x=w+\sigma t$. Since $\sigma>\sqrt{2}$, by the definition of $m(t)$ it holds that $m(t)-B-\sigma t<(\sqrt{2}-\sigma)t<0$, for all $t>1$. Therefore, using the standard Gaussian tail bound,

$$\int_{-\infty}^{-x}\mathrm{e}^{-w^2/2}\frac{\mathrm{d}w}{\sqrt{2\pi}}\leq\frac{1}{\sqrt{2\pi}x}\mathrm{e}^{-x^2/2},\quad x>0, \tag{3.10}$$

we can bound (3.9) using $m^2(t)=2t-3t\log t+\left(3\log t/(2\sqrt{2})\right)^2$ from above by

$$\frac{\sqrt{t}}{\sqrt{2\pi}(B-1+\sigma t-m(t))}\mathrm{e}^{-\sigma m(t)+\frac{\sigma^2t}{2}}\mathrm{e}^{-(m(t)-B+1-\sigma t)^2/2t}\underset{t\uparrow\infty}{\sim}\frac{t}{\sqrt{2\pi}(\sigma-\sqrt{2})}\mathrm{e}^{-t+(\sqrt{2}-\sigma)(B-1)}. \tag{3.11}$$

Next, we analyse the probability in the r.h.s. of (3.8). We bound it, for $B<t^\gamma/3$, from above by

$$\mathbb{P}\left\{\xi(s)\leq2(s\wedge(t-s))^\gamma,\forall s\in[r\vee B^{1/\gamma},(t-B^{1/\gamma})\wedge(t-r)]\right\}. \tag{3.12}$$

By the proof of [2, Theorem 2.3, see (5.55)], for all r large enough, probability (3.12) is bounded from above by

$$\mathbb{P}\left\{\xi(s)\leq0,\forall s\in[r\vee B^{1/\gamma},(t-B^{1/\gamma})\wedge(t-r)]\right\}(1+\epsilon)\leq\frac{2(B^{1/\gamma}\wedge r)}{t-2(B^{1/\gamma}\wedge r)}(1+\epsilon), \tag{3.13}$$

where in the last step we used [2, Lemma 3.4]. Plugging the estimates from (3.11) and (3.13) into (3.8), we get

$$S(B,t)\leq\left(\frac{2(B^{1/\gamma}\vee r)}{t-2(B^{1/\gamma}\vee r)}(1+\epsilon)\mathbb{1}_{\{B>t^\gamma/3\}}+\mathbb{1}_{\{B\leq t^\gamma/3\}}\right)\frac{t\mathrm{e}^{(\sqrt{2}-\sigma)(B-1)}}{\sqrt{2\pi}(\sigma-\sqrt{2})}(1+o(1)). \tag{3.14}$$

Note that in (3.14) and below $o(1)$ denotes a t-dependent non-random quantity with

$$o(1)\underset{t\uparrow\infty}{\longrightarrow}0. \tag{3.15}$$

From (3.14) follows that $\lim_{t\uparrow\infty}\sum_{B>t/3} S(B,t)=0$ and

$$\sum_{B=A+1}^{t^\gamma/3} S(B,t) \le \sum_{B=A+1}^{t^\gamma/3} \frac{2t(B^{1/\gamma}\vee r)\mathrm{e}^{(\sqrt{2}-\sigma)(B-1)}}{\sqrt{2\pi}(\sigma-\sqrt{2})(t-2(B^{1/\gamma}\vee r))}(1+\epsilon), \tag{3.16}$$

which can be made smaller than $\epsilon/2$ by taking A large enough since $\sqrt{B^{1/\gamma}\wedge r}\mathrm{e}^{(\sqrt{2}-\sigma)(B-1)}$ is summable in B (because $\sqrt{2}-\sigma<0$). This concludes the proof of Theorem 2.1 in case (a). □

Case (b). In this case, the analysis is somewhat more intricate and we have to employ the imaginary part of the energy.
Short outline of the proof. To prove (2.1), we first apply the Chebyshev inequality to the absolute value of the truncated partition function. Then, we compute the second moment which arises in the Chebyshev inequality. Along the way, we first use Representation (1.12) and compute the expectation w.r.t. $z(t)$ conditionally on $\mathcal{F}^{\mathbb{T}_t}$, see (3.19). Starting from (3.22), we use the so-called upper envelope for the given path of $x(t)$ (see [2, Theorem 2.2]) to control the expectation w.r.t. $x(t)$. Technically, we have to distinguish between three regimes for the time of the most recent common ancestor $q_{k,l}=d(x_k(t),x_l(t))$. The corresponding terms are controlled separately starting from Eq. (3.35).[4]

Proof of Proposition 2.1 in case (b). We proceed as in case (a) until (3.4). This time, using Chebyshev's inequality, we bound the probability in (3.4) by

$$\frac{1}{\delta^2}\mathbb{E}\left[\left|\sum_{k=1}^{n(t)} \mathrm{e}^{\sigma(x_k(t)-m(t))+i\tau y_k(t)}\mathbb{1}_{\{x_k(t)-m(t)<-A\}\cap\{x_k\in\mathcal{U}_{t,r,\gamma}\}}\right|^2\right], \tag{3.17}$$

We introduce the shorthand notation $\widetilde{x}_k(t):=x_k(t)-m(t)$, $k\le n(t)$. Using this notation, together with Representation (1.12), we get that (3.17) is equal to

$$\frac{1}{\delta^2}\mathbb{E}\left[\left|\sum_{k=1}^{n(t)} \mathrm{e}^{(\sigma+i\rho\tau)x_k(t)-\sigma m(t)+i\sqrt{1-\rho^2}\tau z_k(t)}\mathbb{1}_{\{\widetilde{x}_k(t)<-A\}\cap\{x_k\in\mathcal{U}_{t,r,\gamma}\}}\right|^2\right]. \tag{3.18}$$

Define $\lambda:=\sigma+i\rho\tau$. Observe that $|z|^2=z\bar{z}$, for $z\in\mathbb{C}$. Hence, the expectation in (3.18) is equal to

$$\mathbb{E}\left[\sum_{k,l=1}^{n(t)} \mathrm{e}^{\overline{\lambda}x_l(t)+\lambda x_k(t)-2\sigma m(t)+i\sqrt{1-\rho^2}\tau(z_l(t)-z_k(t))}\mathbb{1}_{\forall_{j\in\{l,k\}}(\{\widetilde{x}_j(t)<-A\}\cap\{x_j\in\mathcal{U}_{t,r,\gamma}\})}\right] \tag{3.19}$$

$$=\mathbb{E}\left[\sum_{k,l=1}^{n(t)}\Big(\mathrm{e}^{\overline{\lambda}x_l(t)+\lambda x_k(t)-2\sigma m(t)}\mathbb{1}_{\forall_{j\in\{l,k\}}(\{\widetilde{x}_j(t)<-A\}\cap\{x_j\in\mathcal{U}_{t,r,\gamma}\})} \tag{3.20}$$
$$\times\,\mathbb{E}\left[\mathrm{e}^{i\sqrt{1-\rho^2}\tau(z_l(t)-z_k(t))}\mid\mathcal{F}^{\mathbb{T}_t}\right]\Big)\right],$$

where we used that $(z_k(t),k\le n(t))$ is, conditionally on $\mathbb{T}_t$, independent from $(x_k(t),k\le n(t))$. Since $(z_k(t),k\le n(t))$ is a BBM on the same GW tree as x, (3.19) is equal to

$$\mathbb{E}\left[\sum_{k,l=1}^{n(t)} \mathrm{e}^{\overline{\lambda}x_l(t)+\lambda x_k(t)-2\sigma m(t)+(1-\rho^2)\tau^2(t-d(x_l(t),x_k(t)))}\mathbb{1}_{\forall_{j\in\{l,k\}}\{\widetilde{x}_j(t)<-A\}\cap\{x_j\in\mathcal{U}_{t,r,\gamma}\}}\right]. \tag{3.21}$$

[4]Note that this approach to control the second moment differs from the one used in [19]. The latter one relies on decomposing the paths of the BBM particles according to the time and location of the minimal position along the given path.

We introduce the time of the most recent common ancestor $q_{k,l} = d(x_k(t), x_l(t))$, where $d(\cdot,\cdot)$ is defined in (1.1), and rewrite (3.21) as $\sum_{B>1} T(B,t)$, where

$$T(B,t) := \mathbb{E}\left[\sum_{k,l=1}^{n(t)} \mathrm{e}^{\overline{\lambda} x_l(t)+\lambda x_k(t)-2\sigma m(t)} \mathrm{e}^{(1-\rho^2)\tau^2(t-q_{k,l})} \mathbb{1}_{\mathcal{U}^{l,k}_{B,q,t}}\right], \tag{3.22}$$

and

$$\begin{aligned}\mathcal{U}^{l,k}_{B,q,t} := \cap_{j\in\{l,k\}} \{\widetilde{x}_j(t) < -A\} \cap \{x_j(s) \le U_{t,\gamma}(s), \forall s \in [r, t-r]\} \\ \cap \{x_j(q_{k,l}) - U_{t,\gamma}(q_{k,l}) \in [-B+1, -B]\}.\end{aligned} \tag{3.23}$$

Similar to (3.7), we now relax conditions on the path of the particle. If $q_{k,l} > \frac{3}{4}t$, then we get

$$\begin{aligned}\mathcal{U}^{l,k}_{B,q,t} \subset \cap_{j\in\{l,k\}} \{\widetilde{x}_j(t) < -A\} \cap \{x_l(q_{k,l},t) - U_{t,\gamma}(q_{k,l}) \in [-B+1,-B]\} \\ \cap \{\xi^q_l(s) \le 8(s \wedge (q_{k,l}-s))^\gamma, \forall s \in [B^{1/\gamma} \vee r, q_{k,l} - (B^{1/\gamma} \wedge r)]\} =: \mathcal{T}^{l,k}_{B,q,t},\end{aligned} \tag{3.24}$$

where $\xi^q_l(s) := x_l(s,t) - \frac{s}{q} x_l(q_{k,l},t)$ is a Brownian bridge from 0 to 0 in time $q_{k,l}$, which is, in particular, independent of $x_l(q_{k,l},t)$. Moreover, for $q \le \frac{3}{4}t$, we have

$$\mathcal{U}^{l,k}_{B,q,t} \subset \cap_{j\in\{l,k\}} \{\widetilde{x}_j(t) < -A\} \cap \{x_l(q_{k,l},t) - U_{t,\gamma}(q_{k,l}) \in [-B+1,-B]\} =: \mathcal{S}^{l,k}_{B,q,t}. \tag{3.25}$$

Hence, $T(B,t)$ defined in (3.22) is bounded from above by

$$\begin{aligned}&\mathbb{E}\left[\sum_{k,l=1}^{n(t)} \mathrm{e}^{\overline{\lambda} x_l(t)+\lambda x_k(t)-2\sigma m(t)} \mathrm{e}^{(1-\rho^2)\tau^2(t-q_{k,l})} \left(\mathbb{1}_{\{q_{k,l}>\frac{3}{4}t\}\cap\mathcal{T}^{l,k}_{B,q,t}} + \mathbb{1}_{\{q_{k,l}\le\frac{3}{4}t\}\cap\mathcal{S}^{l,k}_{B,q,t}}\right)\right] \\ &= K\int_0^t \mathrm{d}q\, \mathrm{e}^{2t-q+(1-\rho^2)\tau^2(t-q)} \int_{U_{t,\gamma}(q)-B}^{U_{t,\gamma}(q)-B+1} \mathrm{d}x \int_{-\infty}^{m(t)-A-x} \mathrm{d}y \int_{-\infty}^{m(t)-A-x} \mathrm{d}y' \\ &\quad\times \mathrm{e}^{\sigma(2x+y+y'-2m(t))+i\rho\tau(y'-y)} \mathrm{e}^{-\frac{y^2+y'^2}{2(t-q)}} \frac{1}{2\pi(t-q)} \mathrm{e}^{-\frac{x^2}{2q}} \frac{1}{\sqrt{2\pi q}} \\ &\quad\times \left(\mathbb{1}_{\{q\le\frac{3}{4}t\}} + \mathbb{1}_{\{q\ge\frac{3}{4}t\}} \mathbb{P}\left\{\xi^q(s) \le 8(s\wedge(q-s))^\gamma, \forall s \in [B^{1/\gamma}\vee r, q - B^{1/\gamma}\wedge r]\right\}\right),\end{aligned} \tag{3.26}$$

where $K = \sum_{k=1}^\infty k(k-1)p_k$. It is in (3.26) that we need the second moment assumption on the distribution $(p_k)_{k\ge0}$, cf. Footnote 3. First, observe that, for $B < t^\gamma/3$, as in (3.13), the probability in (3.26) is bounded from above by $\frac{2(B^{1/\gamma}\vee r)}{q-2(B^{1/\gamma}\vee r)}(1+\epsilon)$. Observe that $m(t) - A - x \le m(t) - A - U_{t,\gamma}(q) + B$. We compute first the integrals with respect to y and y' in (3.26), i.e.,

$$\int_{-\infty}^{\mathcal{D}_{A,B,q}} \int_{-\infty}^{\mathcal{D}_{A,B,q}} \mathrm{e}^{\sigma(2x+y+y'-2m(t))+i\rho\tau(y'-y)} \mathrm{e}^{-\frac{y^2+y'^2}{2(t-q)}} \frac{\mathrm{d}y\mathrm{d}y'}{2\pi(t-q)}, \tag{3.27}$$

where $\mathcal{D}_{A,B,q} := m(t) - A - U_{t,\gamma}(q) + B$. We make the following change of variables

$$y = w + \lambda(t-q) \quad \text{and} \quad y' = w' + \overline{\lambda}(t-q). \tag{3.28}$$

Hence, (3.27) is equal to

$$\mathrm{e}^{2\sigma(x-m(t))+(\sigma^2-(\rho\tau)^2)(t-q)} \int_{-\infty}^{\mathcal{D}_{A,B,q}-\lambda(t-q)} \int_{-\infty}^{\mathcal{D}_{A,B,q}-\overline{\lambda}(t-q)} \mathrm{e}^{-\frac{w^2+w'^2}{2(t-s)}} \frac{\mathrm{d}w\mathrm{d}w'}{2\pi(t-q)}. \tag{3.29}$$

Using (3.10), we bound (3.29) from above by

$$\mathrm{e}^{2\sigma(x-m(t))+(\sigma-\tau^2)(t-q)}\Big(\mathbb{1}_{\{\mathcal{D}_{A,B,q}\geq\sigma(t-q)\}} + \exp\left(-\frac{(\mathcal{D}_{A,B,q}-\lambda(t-q))^2+\big(\mathcal{D}_{A,B,q}-\overline{\lambda}(t-q)\big)^2}{2(t-q)}\right)\mathbb{1}_{\{\mathcal{D}_{A,B,q}\leq\sigma(t-q)\}}\Big). \quad (3.30)$$

Next we carry out the integration over x in (3.26). Note that

$$\int_{U_{t,\gamma}(q)-B}^{U_{t,\gamma}(q)-B+1}\mathrm{e}^{2\sigma x}\mathrm{e}^{-\frac{x^2}{2q}}\frac{\mathrm{d}x}{\sqrt{2\pi q}} = \mathrm{e}^{2\sigma^2 q}\int_{U_{t,\gamma}(q)-B-2\sigma q}^{U_{t,\gamma}(q)-B+1-2\sigma q}\mathrm{e}^{-\frac{v^2}{2q}}\frac{\mathrm{d}v}{\sqrt{2\pi q}}, \quad (3.31)$$

where we made the change of variables $x = v + 2\sigma q$. Observe that $U_{t,\gamma}(q) - 2\sigma q \leq (\sqrt{2} - 2\sigma)q < 0$, since $\sigma \geq \frac{1}{\sqrt{2}}$. Therefore, using (3.10), the right-hand side of (3.31) is bounded from above by

$$\frac{\sqrt{q}}{2\sigma q - U_{t,\gamma}(q) + B}\mathrm{e}^{2\sigma^2 q}\mathrm{e}^{-(U_{t,\gamma}(q)-B-2\sigma q)^2/2q}. \quad (3.32)$$

Using the bounds (3.32) and (3.30) in (3.26), we get that (3.26) is bounded from above by

$$\begin{aligned} &K\int_0^t \frac{\sqrt{q}\mathrm{e}^{2t-q+2\sigma^2 q}\mathrm{e}^{-(U_{t,\gamma}(q)-B-2\sigma q)^2/2q}}{2\sigma q - U_{t,\gamma}(q) + B}\mathrm{e}^{-2\sigma m(t)+(\sigma^2-\tau^2)(t-q)} \\ &\quad\times\Big(\mathbb{1}_{\{\mathcal{D}_{A,B,q}\geq\sigma(t-q)\}} + \mathrm{e}^{-\frac{\left(\mathcal{D}_{A,B,q}-\lambda(t-q)\right)^2+\left(\mathcal{D}_{A,B,q}-\overline{\lambda}(t-q)\right)^2}{2(t-q)}}\mathbb{1}_{\{\mathcal{D}_{A,B,q}\geq\sigma(t-q)\}}\Big) \\ &\quad\times\Big(\mathbb{1}_{\{q\leq\frac{3}{4}t\}} + \mathbb{1}_{\{q\geq\frac{3}{4}t,\ B<t^\gamma/3\}}\frac{2(B^{1/\gamma}\vee r)}{q-2(B^{1/\gamma}\vee r)}(1+\epsilon)\Big)\,\mathrm{d}q. \end{aligned} \quad (3.33)$$

Using that $U_{t,\gamma}(q) - 2\sigma q = (\sqrt{2} - 2\sigma)q - \frac{q}{t}\frac{3}{2\sqrt{2}}\log t + (q \wedge (t-q))^\gamma$, we start to simplify (3.33). We get

$$\begin{aligned} &\mathrm{e}^{2t-q}\mathrm{e}^{2\sigma^2 q}\mathrm{e}^{-(U_{t,\gamma}(q)-B-2\sigma q)^2/2q}\mathrm{e}^{-2\sigma m(t)+\frac{(\sigma^2-\tau^2)(t-q)}{2}} \\ &\underset{t\uparrow\infty}{\sim}\mathrm{e}^{(t-q)\left((\sigma-\sqrt{2})^2-\tau^2\right)+\left(\frac{3\sigma}{\sqrt{2}}+\frac{(\sqrt{2}-2\sigma)3q}{2\sqrt{2}t}\right)\log t-(\sqrt{2}-2\sigma)(q\wedge(t-q))^\gamma+(\sqrt{2}-2\sigma)B}. \end{aligned} \quad (3.34)$$

Note that by assumption on σ and τ we have $(\sigma - \sqrt{2})^2 - \tau^2 < 0$ and $\sqrt{2} - 2\sigma < 0$. Cutting the domain of integration in (3.33) into three parts $q \in [0, t-\log(t)^\alpha]$, $q \in (t-\log(t)^\alpha, t-\frac{A}{2}]$ and $q \in (t-\frac{A}{2}, t]$, for some fixed $\alpha > 1$, we get the following three terms

$$K\int_0^t \ldots\mathrm{d}q = K\left(\int_0^{t-\log(t)^\alpha} + \int_{t-\log(t)^\alpha}^{t-\frac{A}{2}} + \int_{t-\frac{A}{2}}^{t}\right)\ldots\mathrm{d}q =: K\left((\mathrm{I1}) + (\mathrm{I2}) + (\mathrm{I3})\right). \quad (3.35)$$

We bound (I1) from above by

$$\begin{aligned} &\int_0^{t-\log(t)^\alpha}\mathrm{e}^{(t-q)\left((\sigma-\sqrt{2})^2-\tau^2\right)+\left(\frac{(\sqrt{2}-2\sigma)3q}{2\sqrt{2}t}+\frac{3\sigma}{\sqrt{2}}\right)\log t-(\sqrt{2}-2\sigma)(q\wedge(t-q))^\gamma+(\sqrt{2}-2\sigma)B}\mathrm{d}q(1+o(1)) \\ &\leq \mathrm{e}^{(\sqrt{2}-2\sigma)B+\frac{3\sigma}{\sqrt{2}}\log t}\int_0^{t-\log(t)^\alpha}\mathrm{e}^{(t-q)\left((\sigma-\sqrt{2})^2-\tau^2\right)-(\sqrt{2}-2\sigma)(q\wedge(t-q))^\gamma}\mathrm{d}q(1+o(1)) \\ &\leq \mathrm{e}^{(\sqrt{2}-2\sigma)B}\mathrm{e}^{C\log(t)^\alpha\left((\sigma-\sqrt{2})^2-\tau^2\right)+\frac{3\sigma}{\sqrt{2}}\log t-(\sqrt{2}-2\sigma)\log(t)^{\gamma\alpha}}, \quad t\uparrow\infty, \end{aligned} \quad (3.36)$$

for some constant $C > 0$. Hence,

$$K\sum_{B>1}(\mathrm{I1}) \leq K\mathrm{e}^{C\log(t)^\alpha\left((\sigma-\sqrt{2})^2-\tau^2\right)+\frac{3\sigma}{\sqrt{2}}\log t-(\sqrt{2}-2\sigma)\log(t)^{\gamma\alpha}}\sum_{B>1}\mathrm{e}^{(\sqrt{2}-2\sigma)B}, \quad (3.37)$$

since $\sqrt{2}-2\sigma<0$, we have $\sum_{B>1}\mathrm{e}^{(\sqrt{2}-2\sigma)B}<\infty$. Hence, we can choose t_0 such that, for all $t>t_0$, the r.h.s. of (3.37) less than $\frac{\epsilon}{6}$. For $q\in(t-\log(t)^\alpha,t]$, we observe first that

$$\mathrm{e}^{\left(\frac{(\sqrt{2}-2\sigma)3q}{2\sqrt{2}t}+\frac{3\sigma}{\sqrt{2}}\right)\log t}\underset{t\uparrow\infty}{\sim}\mathrm{e}^{\frac{3}{2}\log t}, \tag{3.38}$$

and, moreover,

$$\frac{2\sqrt{q}(B^{1/\gamma}\vee r)}{(2\sigma q-U_{t,\gamma}(q)+B)\left(q-2(B^{1/\gamma}\vee r)\right)}\le C'\frac{2(B^{1/\gamma}\vee r)}{\sqrt{t}(t-2(B^{1/\gamma}\vee r))}, \tag{3.39}$$

for some constant $C'>0$. Using (3.38) and (3.39), we bound (I2) from above by

$$\begin{aligned}&\int_{t-\log(t)^\alpha}^{t-\frac{A}{2}}\mathrm{e}^{(t-q)\left((\sigma-\sqrt{2})^2-\tau^2\right)-(\sqrt{2}-2\sigma)(t-q)^\gamma+(\sqrt{2}-2\sigma)B}C't\mathrm{d}q\\&\qquad\qquad\times\left(\tfrac{2(B^{1/\gamma}\vee r)}{(t-2(B^{1/\gamma}\vee r))}\mathbb{1}_{\{B<t^\gamma/3\}}+\mathbb{1}_{\{B\ge t^\gamma/3\}}\right)(1+o(1))\\&\quad\le C_2\mathrm{e}^{\frac{A}{2}((\sigma-\sqrt{2})^2-\tau^2)}\mathrm{e}^{(\sqrt{2}-2\sigma)B}\left((B^{1/\gamma}\vee r)\mathbb{1}_{\{B<t^\gamma/3\}}+t\mathbb{1}_{\{B\ge t^\gamma/3\}}\right)(1+o(1)),\end{aligned} \tag{3.40}$$

as $t\uparrow\infty$. Using (3.40), we get that $K\sum_{B>1}$(I2) is bounded from above by

$$KC_2\mathrm{e}^{\frac{A}{2}((\sigma-\sqrt{2})^2-\tau^2)}\left(\sum_{B=1}^{[t^\gamma/3]}\mathrm{e}^{(\sqrt{2}-2\sigma)B}(B^{1/\gamma}\vee r)+\sum_{B>[t^\gamma/3]}\mathrm{e}^{(\sqrt{2}-2\sigma)B}t\right)(1+o(1)), \tag{3.41}$$

as $t\uparrow\infty$. Again, since $2-2\sigma<0$, we have $\sum_{B>1}B^{\frac{1}{\gamma}}\mathrm{e}^{(\sqrt{2}-2\sigma)B}<\infty$ and $(\sigma-\sqrt{2})^2-\tau^2<0$. Hence, there exist t_1 and A_1 such that, for all $t>t_1$ and all $A>A_1$, we have that (3.41)$\le\frac{\epsilon}{6}$. Since $\mathcal{D}_{A,B,q}-\sigma(t-q)<0$ for $t-q\le\frac{A}{\sqrt{2}}$ and $B\le\frac{A}{2}$, we bound (I3) from above by

$$\begin{aligned}&\int_{t-\frac{A}{2}}^{t}\mathrm{e}^{(t-q)\left((\sigma-\sqrt{2})^2-\tau^2\right)}\mathrm{e}^{-(\sqrt{2}-2\sigma)(t-q)^\gamma+(\sqrt{2}-2\sigma)B}C't\left(\tfrac{2(B^{1/\gamma}\vee r)}{(t-2(B^{1/\gamma}\vee r))}\mathbb{1}_{\{B<t^\gamma/3\}}+\mathbb{1}_{\{B\ge t^\gamma/3\}}\right)\\&\times\left(\mathbb{1}_{\{B<\frac{A}{2}\}}\mathrm{e}^{-\frac{((1-\sqrt{2}\sigma)A)^2}{(t-q)}}(1+o(1))+\mathbb{1}_{\{B\ge\frac{A}{2}\}}\right)\mathrm{d}q,\quad t\uparrow\infty.\end{aligned} \tag{3.42}$$

Using that $(\sigma-\sqrt{2})^2-\tau^2<0$ and $\sqrt{2}-2\sigma<0$, we bound (3.42) from above by

$$\begin{aligned}&\int_{t-\frac{A}{2}}^{t}\mathrm{e}^{-(\sqrt{2}-2\sigma)(\frac{A}{2})^\gamma+(\sqrt{2}-2\sigma)B}\tilde{C}\left(\mathbb{1}_{\{B<t/3\}}2(B^{1/\gamma}\wedge r)+t\mathbb{1}_{\{B\ge t/3\}}\right)\\&\quad\times\left(\mathbb{1}_{\{B<\frac{A}{2}\}}\mathrm{e}^{-\frac{((1-\sqrt{2}\sigma)A)^2}{A/2}}(1+o(1))+\mathbb{1}_{\{B\ge\frac{A}{2}\}}\right)\mathrm{d}q\\&\le\frac{A}{2}\mathrm{e}^{-(\sqrt{2}-2\sigma)(\frac{A}{2})^\gamma+(\sqrt{2}-2\sigma)B}\tilde{C}\left(\mathbb{1}_{\{B<t^\gamma/3\}}2(B^{1/\gamma}\wedge r)+t\mathbb{1}_{\{B\ge t^\gamma/3\}}\right)\\&\quad\times\left(\mathbb{1}_{\{B<\frac{A}{2}\}}\mathrm{e}^{-\frac{((1-\sqrt{2}\sigma)A)^2}{A/2}}(1+o(1))+\mathbb{1}_{\{B\ge\frac{A}{2}\}}\right),\quad t\uparrow\infty.\end{aligned} \tag{3.43}$$

Using (3.43), together with the fact that, for all $t>\frac{3A^\gamma}{2}$, it holds that $\frac{t^\gamma}{3}>\frac{A}{2}$, we get that, for all such t, the sum $K\sum_{B>1}$(I3) is bounded from above by

$$\begin{aligned}&K\tilde{C}\frac{A}{2}\mathrm{e}^{-(\sqrt{2}-2\sigma)\left(\frac{A}{2}\right)^\gamma}\Bigg(\sum_{B>1}^{A/2}\mathrm{e}^{(\sqrt{2}-2\sigma)B}\mathrm{e}^{-\frac{2\left((1-\sqrt{2}\sigma)A\right)^2}{A}}(B^{1/\gamma}\vee r)\\&\quad+\sum_{B>A/2}^{t^\gamma/3}\mathrm{e}^{(\sqrt{2}-2\sigma)B}(B^{1/\gamma}\vee r)+\sum_{B>t^\gamma/3}t\mathrm{e}^{(\sqrt{2}-2\sigma)B}\Bigg)(1+o(1)),\quad t\uparrow\infty.\end{aligned} \tag{3.44}$$

Hence, there exist t_2 and A_2 such that for all $t > t_2$ and $A > A_2$ the term in (3.44) is not greater than $\frac{\epsilon}{6}$. Now, combining the bounds in (3.37), (3.41) and (3.44), we get that, for all $t > \max\{t_0, t_1, t_2\}$ and $A > \max\{A_1, A_2\}$, $\sum_{B\geq 1} T(B,t) \leq \frac{\epsilon}{6} + \frac{\epsilon}{6} + \frac{\epsilon}{6} = \frac{\epsilon}{2}$. By (3.4), this concludes the proof of Proposition 2.1. □

A Martingale convergence

For $\beta = \sigma + i\tau$, set $M_\beta(t) := \mathrm{e}^{-t\left(1+\frac{\sigma^2}{2}-\frac{\tau^2}{2}+i\rho\tau\right)} \sum_{k=1}^{n(t)} \mathrm{e}^{\sigma x_k(t)+i\tau y_k(t)}$.

Proposition A.1. *For $\beta \in \mathbb{C}$ with $|\beta| < 1$, $M_\beta(t)$ is an L^2-bounded martingale with expectation one. In particular, $M_\beta(t)$ converges to a non-degenerate limit M_β a.s. and in L^2 as t tends to infinity.*

Proof. Using Representation (1.12), one easily verifies that $\mathbb{E}[M_\beta(t)] = 1$ and that it is indeed a martingale. It remains to show the L^2-boundedness of $M_\beta(t)$. We have

$$\mathbb{E}\left[|M_\beta(t)|^2\right] = \mathrm{e}^{-2t\left(1+\frac{\sigma^2}{2}-\frac{\tau^2}{2}\right)} \mathbb{E}\left[\sum_{k,l=1}^{n(t)} \mathrm{e}^{\sigma(x_k(t)+x_l(t))+i\tau(y_k(t)-y_l(t))}\right]. \tag{A.1}$$

Using Representation (1.12), we rewrite the right-hand side of (A.1) as

$$\mathrm{e}^{-2t\left(1+\frac{\sigma^2}{2}-\frac{\tau^2}{2}\right)} \mathbb{E}\left[\sum_{k,l=1}^{n(t)} \mathrm{e}^{\bar{\lambda} x_l(t)+\lambda x_k(t))+i\tau(1-\rho^2)(z_k(t)-z_l(t))}\right], \tag{A.2}$$

where $\lambda = \sigma + i\rho\tau$ and $(z_k(t))_{k\leq n(t)}$ are the particles of a BBM on $\mathbb{T}_t$ that is independent from $X(t)$. By conditioning on $\mathcal{F}^{\mathbb{T}_t}$ as in (3.19), we have that (A.2) is equal to

$$\mathrm{e}^{-2t\left(1+\frac{\sigma^2}{2}-\frac{\tau^2}{2}\right)} \mathbb{E}\left[\mathrm{e}^{-(1-\rho^2)\tau^2(t-d(x_k(t),x_l(t)))} \sum_{k,l=1}^{n(t)} \mathrm{e}^{\bar{\lambda} x_l(t)+\lambda x_k(t)}\right]. \tag{A.3}$$

Similarly to (3.26), the expectation in (A.3) is equal to

$$\begin{aligned} &K \int_0^t \mathrm{d}q \mathrm{e}^{2t-q-(1-\rho^2)\tau^2(t-q)} \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{\sqrt{2\pi q}} \int_{-\infty}^{\infty} \frac{\mathrm{d}y}{\sqrt{2\pi(t-q)}} \\ &\times \int_{-\infty}^{\infty} \frac{\mathrm{d}y'}{\sqrt{2\pi(t-q)}} \mathrm{e}^{2\sigma x+\sigma(y+y')+i\tau\rho(y-y')} \mathrm{e}^{-\frac{y^2+y'^2}{2}} \mathrm{e}^{-x^2/2}. \end{aligned} \tag{A.4}$$

Computing first the integrals with respect to y and y', we get that (A.4) is equal to

$$\begin{aligned} &K \int_0^t \mathrm{d}q \mathrm{e}^{2t-q-(1-\rho^2)\tau^2(t-q)+(\sigma^2-\rho^2\tau^2)(t-q)} \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{\sqrt{2\pi q}} \mathrm{e}^{2\sigma x} \mathrm{e}^{-x^2/2} \\ &= K \int_0^t \mathrm{d}q \mathrm{e}^{2t-q-\tau^2(t-q)+\sigma^2(t-q)} \mathrm{e}^{2\sigma^2 q}. \end{aligned} \tag{A.5}$$

Plugging (A.5) back into (A.3), we get that (A.3) is equal to

$$\mathrm{e}^{-2t\left(1+\frac{\sigma^2}{2}-\frac{\tau^2}{2}\right)} K \int_0^t \mathrm{d}q \mathrm{e}^{2t-q-\tau^2(t-q)+\sigma^2(t-q)} \mathrm{e}^{2\sigma^2 q} = K \int_0^t \mathrm{d}q \mathrm{e}^{q(\sigma^2+\tau^2-1)} \leq C, \tag{A.6}$$

for some constant $C > 0$ uniformly in t since $\sigma^2 + \tau^2 < 1$ by assumption. Hence, $M_\beta(t)$ is an L^2-bounded martingale with expectation one and converges as $t \uparrow \infty$ to a non-degenerate limit a.s. and in L^2.

□

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